F3HX 12
Maths: Technician 1

H7K0 33
Engineering Maths 1

Trigonometric Waves
Supplement

Engineering, Science and Technology
Trigonometric Waves - Supplement

These notes deal with radian measure waves only.

So far the wave functions that you have seen have, more or less, been confined to the type \( y = R \sin(\omega t + \alpha) \) or \( y = R \cos(\omega t + \alpha) \) where \( \omega \) is a whole number or a whole number fraction.

Examples include \( y = 5 \sin(2t + 0.8) \), and \( y = 3 \cos(0.5t - 0.4) \).

Although the functions look relatively simple they lead to unwieldy geometry.

\[ y = 5 \sin(2t + 0.8) \] has a period of \( P = \frac{2\pi}{2} = \pi \approx 3.142 \)

\[ y = 3 \cos(0.5t - 0.4) \] has a period of \( P = \frac{2\pi}{0.5} = 4\pi \approx 12.57 \)

The problem is that \( \pi \) is unmeasurable but most waves you see on an oscilloscope or elsewhere have a fairly easily understood period such as \( 4 \text{ ms} \ (4 \times 10^{-3} \text{ seconds}) \).
A Different Approach:

Although the word circle will be used, try to imagine any circular spinning object such as a pulley or a wheel. (spinning anti-clockwise of course)

We have to go back to the definition of the radian.

What do we mean by a radian?

There are two points of view:

1. The view from the centre of the circle and 2. the view from the outside of the circle

1. Observed from the centre a radian is the angle of turn $\theta = \frac{180}{\pi} \approx 57.3^\circ$.

2. Observed from the circumference a radian is a distance of 1 radius around the outside of the circle. Altogether a distance of $2\pi$ radii have to be travelled to make one complete turn (that is because $C = 2\pi r$)

We will leave the observer at the centre of the circle, with their degree measure protractor, and concentrate on the observer with their (bendy) ruler at the outside of the circle.
What will this observer see when the circle spins at 1 cycle per second (1 Hz)?
They will measure $2\pi$ radii per second.

We seek to model an oscilloscope trace mathematically. An oscilloscope is plotting amplitude against time. Mathematically the amplitude at a time $t$ is the maximum amplitude multiplied by the sine of the angle (measured in radians) at that time. All graphs will have time in seconds on the horizontal axis and amplitude on the vertical axis (as usual). We should be able to see exactly the same graph that would be displayed on an oscilloscope.

In $y = R \sin(\omega t + \alpha)$ the part in brackets must be measured in radians but $t$ is measured in seconds.
So, $\omega$ is measured in $\frac{rad}{s}$ which is the angular velocity.

Period $P = \frac{2\pi}{\omega}$. for the 1 Hz wave $\lambda = 1$ therefore $1 = \frac{2\pi}{\omega}$ and $\omega = 2\pi$ or from the definition above $\omega = 2\pi f$ therefore $\omega = 2\pi \times 1 = 2\pi$.

If the circle is spinning at a of 2 cycles per second (a frequency of 2 hertz) then $\omega = 2\pi \times 2 = 4\pi$ rad/s

$\omega = 2\pi f$ where $f$ is the frequency in turns per second or hertz.
A frequency of 50 Hz will make $\omega = 2\pi \times 50 = 100\pi$ rad/second.

The phase angle $\alpha$ is measured in radians.
Phase shift is measured in seconds

$$\phi = \frac{\alpha \text{rad}}{\omega} = \frac{\alpha}{\omega} \left(\frac{\text{rad}}{s}\right) = \frac{\text{rad}}{\text{s}}$$

A wave of amplitude 5 and period of 1 second would then have the equation

$$y = 5 \sin(2\pi t)$$

From the perspective of an observer at the circumference it is going at 1 cycle a second (the graph) and $2\pi$ radii per second (the equation).
Phase Shift and Phase Angle

\[ y = R \sin(\omega t + \alpha) \]
where angular velocity \( \omega = 2\pi f \) and the phase angle \( = \alpha \)

The phase shift \( \phi = \frac{\alpha}{\omega} \)

For a measurable leading phase shift of 0.2 seconds when \( \omega = 2\pi \) we will have to put \( 0.2 = \frac{\alpha}{2\pi} \) making \( \alpha = 0.4\pi \) rad. (Alternatively \( y = 5\sin(2\pi t + 0.2) \))

The equation will be \( y = 5\sin(2\pi t + 0.4\pi) \) (Amplitude 5 and period 1 s).

With thanks to Kalid Azad. Please refer to the article below


and Brian Christie, Aberdeen College.
Examples and Exercises

Example 1

Sketch 1 wave of \( y = 20 \cos(4 \pi t) \) and \( y = 20 \cos(4 \pi t + 0.2 \pi) \) on the same axes.

Solution:

\[
\omega = 4 \pi, \text{ Phase angle } = 0.2 \pi
\]

Amplitude \( A = 20 \) \hspace{1cm} \text{Period } \( P = \frac{2 \pi}{4 \pi} = 0.5 \)

Phase shift \( \phi = \frac{0.2 \pi}{4 \pi} = 0.05 \) (Leading by 0.05 seconds)

\[ y = 20 \cos(4 \pi t) \]  \hspace{1cm} \[ y = 20 \cos(4 \pi t + 0.2 \pi) \]
Example 2.

Find the equations of these oscilloscope traces. Wave 1 is the reference wave with wave 2 being out of phase (and leading). Horizontally each division (1 cm) is 0.02 seconds and vertically each division is 1.25 volts.

Note: 0 has been chosen so that we are looking at a sine wave - it is not marked on the screen.

Solution:

**Wave 1** \( V = R \sin(\omega t) \)

- Amplitude 10 divisions \( R = 1.25 \times 4 = 5 \text{ volts} \)
- Period 10 divisions \( P = 0.02 \times 10 = 0.2 \text{ seconds} \)

\[
0.2 = \frac{2\pi}{\omega} \quad \text{Period} \\
0.2 \omega = 2 \pi \quad \omega = \frac{2\pi}{0.2} \quad \omega = 10\pi
\]

\[V = 5 \sin(10\pi t)\]

**Wave 2** \( V = R \sin(\omega t + \alpha) \)

- Phase shift \( \frac{1}{2} \) divisions so looks like
- \( \phi = 0.03 \text{ seconds} \) leading

\[
\phi = 0.03 = \frac{\alpha}{10\pi} \quad 0.03 \times 10\pi = \alpha \quad 0.3\pi = \alpha
\]

\[V = 5 \sin(10\pi t + 0.3\pi)\]
Example 2. Alternative solution method

Find the equations of these oscilloscope traces. Wave 1 is the reference wave with wave 2 being out of phase (and leading).
Horizontally each division (1 cm) is 2 milli-seconds and vertically each division is 1.25 volts.

Note: 0 has been chosen so that we are looking at a sine wave - it is not marked on the screen.

Solution:

Wave 1 \( V = R \sin(\omega t) \)

- Peak to Peak 20 cm = 8 divisions
- Peak to Peak voltage = \( 8 \times 1.25 = 10 \) volts
- Peak voltage \( R = \frac{10}{2} = 5 \) volts
- Period 10 cm = 10 divisions
  \( P = 10 \times 2 \times 10^{-3} = 0.2 \) seconds
  \( f = \frac{1}{P} = \frac{1}{0.2} = 5 \) Hz
  \( \omega = 2\pi f = 2\pi \times 5 = 10\pi \)

\[ V = 5 \sin(10\pi t) \]

Wave 2 \( V = R \sin(\omega (t + \alpha)) \)

- \( V = R \sin(\omega (t + \phi)) \) where \( \phi = \) phase shift in seconds
- Phase shift looks like 1.5 cm, so 0.03 leading
- \( V = 5 \sin\left(10\pi (t + 0.03)\right)\)
- \( V = 5 \sin\left(10\pi t + 10\pi \times 0.03\right)\)

\[ V = 5 \sin\left(10\pi t + 0.3\pi\right) \]
Example 3 - 3 phases of a mains voltage

Peak voltage = \(230 \times \sqrt{2} = 325 \, V\)

\(f = 50 \, Hz\) therefore \(\omega = 2 \pi \times 50 = 100 \pi\)

In terms of degrees we would say that the other two phases are at \(\pm 120^\circ\). This will equate to \(\pm \frac{2 \pi}{3}\) in radian measure.

The equations are \(V = 325 \sin (100 \pi t)\)

and for one phase \(V = 325 \sin \left(100 \pi t + \frac{2 \pi}{3}\right)\)

and the other \(V = 325 \sin \left(100 \pi t - \frac{2 \pi}{3}\right)\).

Note that the phase shift observed will be

\(\phi = \frac{2 \pi}{3} = 0.0067 \text{ seconds} \quad (\phi = \pm 0.0067 \text{ seconds}).\)
Example 4

Solve $V=5\sin(10\pi t+0.3\pi)$ for $0<t<1$ when $V=4$ (this is wave 2 in Example 2)

\[
\begin{align*}
5\sin(10\pi t+0.3\pi) & = 4 \\
\sin(10\pi t+0.3\pi) & = 0.8 \\
10\pi t+0.3\pi & = \sin^{-1}(0.8)
\end{align*}
\]

$10\pi t+0.3\pi = 0.9273, \quad \pi - 0.9273 = 2.2143$

Does this agree with the $y=\sin(x)$ graph?

\[
\begin{align*}
10\pi t & = 0.9273 - 0.3\pi, \quad 2.2143 - 0.3\pi \\
10\pi t & = -0.0152, \quad 1.2718 \\
t & = -0.0152 \cdot \frac{10\pi}{10\pi}, \quad 1.2718 \cdot \frac{10\pi}{10\pi} \\
t & = (-0.0005), \quad 0.0405
\end{align*}
\]

Add period ($P=0.2$)

\[
\begin{align*}
t & = 0.1995, \quad 0.2405 \\
t & = 0.3995, \quad 0.4405 \\
t & = 0.5995, \quad 0.6405 \\
t & = 0.7995, \quad 0.8405 \\
t & = 0.9995, \quad (1.0405)
\end{align*}
\]

Check:

\[
5\sin(10\pi \times 1.0405 + 0.3\pi) = 3.998 \quad \text{an excellent result!}
\]
Exercise.

1. Sketch and label one wave of \( y = 5 \sin(20 \pi t) \) and \( y = 5 \sin(20 \pi t - 0.2 \pi) \) on the same axes.

2. Sketch and label \( V = 12 \cos(50 \pi t) \) and \( V = 10 \cos(50 \pi t - 0.4 \pi) \) on the same axes.

3. Estimate the equations of the vibrations shown. Wave 1 is the principal wave.

4. Estimate the equations of the voltages shown. Wave 1 is the principal wave

Each horizontal division (of 1 cm on the screen) represents 0.01 seconds
Each vertical division (of 1 cm on the screen) represents 1 mm

An Electrical Example

Each horizontal division (of 1 cm on the screen) represents 0.005 seconds
Each vertical division (of 1 cm on the screen) represents 2.5 volts.

A Mechanical Example
5. $X = 6 \sin (20\pi t + 0.4\pi)$

Find the times between which $X$ exceeds 4.2.

(You will first find two adjacent values for $t$ when $6 \sin (20\pi t + 0.4\pi) = 4.2$ and then, be able to put a value to how often per second $X$ exceeds 4.2.)

6. $V = 8 \cos (50\pi t - \frac{3\pi}{4})$

Solve for $0 \leq t \leq 0.1$ when $V = 6$. 

-----------------------------------------------------------------------------------------------------------------------
Answers

1. Wave 1 is \( y = 6 \sin(20\pi t) \)
   Wave 2 is \( y = 6 \sin(20\pi t + 0.4\pi) \)

2. Wave 1 is \( y = 10 \sin(50\pi t) \)
   Wave 2 is \( y = 8 \sin\left(50\pi t + \frac{\pi}{4}\right) \) (or as near as you can, especially the voltage)

3. \( t = -0.007659, 0.01766 \)
   Period is \( P = \frac{2\pi}{20\pi} = 0.1 \text{ seconds} \)
   Frequency is \( f = \frac{1}{P} = \frac{1}{0.1} = 10 \text{ Hz} \)
   The values for \( t \) repeat every 0.1 seconds or the value of \( X \) exceeds 4.2 10 times a second

4. \( t = 0.0204, 0.0296, 0.0604, 0.0696 \)