Engineering Maths 2
H7K1 34

Trigonometric and Hyperbolic Functions

Inverse Trig ratios
Compound Angle Formulae
Trigonometric identities
Hyperbolic functions
Hyperbolic identities
### Trigonometric Ratios

<table>
<thead>
<tr>
<th>Trigonometric Ratio</th>
<th>Definition</th>
<th>Aide memoire</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>sine</strong> (\sin(\alpha))</td>
<td>[ \frac{\text{opposite}}{\text{hypotenuse}} ]</td>
<td><strong>Look at the 3rd letter</strong>&lt;br&gt;coSecant – 1 over sine&lt;br&gt;seCant – 1 over cosine&lt;br&gt;coTangent – 1 over tangent</td>
</tr>
<tr>
<td><strong>cosine</strong> (\cos(\alpha))</td>
<td>[ \frac{\text{adjacent}}{\text{hypotenuse}} ]</td>
<td><strong>Aide memoire:</strong>&lt;br&gt;(\frac{1}{\sin(\alpha)}) (\frac{1}{\cos(\alpha)}) (\frac{1}{\tan(\alpha)})</td>
</tr>
<tr>
<td><strong>tangent</strong> (\tan(\alpha))</td>
<td>[ \frac{\text{opposite}}{\text{adjacent}} ]</td>
<td>cotangent (\frac{1}{\tan(\alpha)})</td>
</tr>
</tbody>
</table>

\[
\sin(\alpha) = \frac{\text{opposite}}{\text{hypotenuse}} \\
\cos(\alpha) = \frac{\text{adjacent}}{\text{hypotenuse}} \\
\tan(\alpha) = \frac{\sin(\alpha)}{\cos(\alpha)} = \frac{\text{opposite}}{\text{adjacent}}
\]

\[
cosec(\alpha) = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{1}{\sin(\alpha)} \\
sec(\alpha) = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{1}{\cos(\alpha)} \\
cot(\alpha) = \frac{\cos(\alpha)}{\sin(\alpha)} = \frac{1}{\tan(\alpha)}
\]
Thanks to Jonathan Fowler

Some interesting questions arise:

1. Prove that \((R \tan(\theta))^2 + R^2 = (R \sec(\theta))^2\)

2. Prove that \((R \csc(\theta))^2 - (R \cot(\theta))^2 = R^2\)

3. Prove that \((R \csc(\theta))^2 + (R \sec(\theta))^2 = (R \tan(\theta) + R \cot(\theta))^2\)
Trigonometric Identities

The unit circle $x^2 + y^2 = 1$ is the generator for the sine and cosine functions.

Hence, $(\cos(x))^2 + (\sin(x))^2 = 1$

or, as normally written

$\cos^2(x) + \sin^2(x) = 1$

Most problems involving trig identities require you to use the relationship above and/or

$\tan(x) = \frac{\sin(x)}{\cos(x)}$

It is also worth knowing that trig functions are complex functions (as in complex numbers). You may have noticed that they can be expressed as a combination of exponential functions as below:

$\sin(x) = \frac{e^{ix} - e^{-ix}}{j2}$ \quad $\cos(x) = \frac{e^{ix} + e^{-ix}}{2}$ \quad where $j^2 = -1$

The assumption must be made, unless you are told otherwise, that all angles are measured in radians so look out for degree symbols $x^o$ and mark your calculator to ensure that you know whether you are calculating in degrees or radians.

Some to try:

Exercise 1

1. Evaluate
   a) $5 \csc(3.2)$ \quad b) $3 \sec(1.37)$ \quad c) $12 \cot(1.97)$

2. Simplify
   a) $3 \sec(x) \sin(x)$ \quad b) $\tan(10\pi t) \csc(10\pi t)$ \quad c) $\sin^2(x) \csc(x)$

3. Simplify
   a) $\tan^2(x) \cos^2(x) + \cot^2(x) \sin^2(x)$ \quad b) $\frac{\sin(x) - \sin^3(x)}{\sin(x)}$ \quad c) $1 - \frac{1}{\sec^2(x)}$
Differentiating annoying trig functions

Example:

\[ y = 5 \cosec(3t) \]. Find the gradient of \( y \) at \( t = 4 \)

All these functions need the Chain Rule to solve them

\[ y = 5 \cosec(3t) \]
\[ y = \frac{5}{\sin(3t)} \]
\[ y = 5\left(\sin(3t)\right)^{-1} \]
\[ y = 5u^{-1} \quad u = \sin(3t) \]

\[ \frac{dy}{du} = -5u^{-2} \quad \frac{du}{dt} = 3\cos(3t) \]

\[ \frac{dy}{dt} = -\frac{5}{u} \times 3\cos(3t) \]
\[ \frac{dy}{dt} = -\frac{5}{\left(\sin(3t)\right)^2} \times 3\cos(3t) \]

\[ \frac{dy}{dt} = -\frac{15\cos(3t)}{(\sin(3t))^2} \]

when \( t = 4 \)

\[ \frac{dy}{dt} = -\frac{15\cos(3 \times 4)}{(\sin(3 	imes 4))^2} \]
\[ \frac{dy}{dt} = -43.96 \]

Now try these:

Exercise 2

1. Find the rate of change of \( 7 \sec(2t) \) when \( t = 1.5 \)

2. Determine the gradient of \( 12 \cot(2\pi t) \) when \( t = 2.43 \)
Multiple Angle Formulae

Using the unit circle (radius = 1) the cosine rule and vectors:

\[ x^2 = 1^2 + 1^2 - 2 \times 1 \times 1 \times \cos(A - B) \]

\[ x^2 = 2 - 2 \cos(A - B) \quad (1) \]

Also, as a vector in rectangular form:

\[ \begin{align*}
\vec{x} &= \begin{bmatrix} \cos(A) - \cos(B) \\ \sin(A) - \sin(B) \end{bmatrix} \\
\end{align*} \]

\[ x^2 = (\cos(A) - \cos(B))^2 + (\sin(A) - \sin(B))^2 \]

\[ x^2 = \cos^2(A) - 2 \cos(A) \cos(B) + \cos^2(B) + \sin^2(A) - 2 \sin(A) \sin(B) + \sin^2(B) \]

\[ \sin^2(A) + \cos^2(A) = 1 \quad \text{and} \quad \sin^2(B) + \cos^2(B) = 1 \quad \text{(diagram above and Pythagoras)} \]

\[ x^2 = 2 - 2 \cos(A) \cos(B) - 2 \sin(A) \sin(B) \quad (2) \]

Putting (1) and (2) together:

\[ 2 - 2 \cos(A - B) = 2 - 2 \cos(A) \cos(B) - 2 \sin(A) \sin(B) \]

\[ -2 \cos(A - B) = -2 \cos(A) \cos(B) - 2 \sin(A) \sin(B) \]

\[ \cos(A - B) = \cos(A) \cos(B) + \sin(A) \sin(B) \]
To arrive at \( \sin(A+B) \) firstly we have (in degrees so that you can visualise it):

\[
\sin(A+B) = \cos(A+B-90^\circ) = \cos(B+A-90^\circ) = \cos(B-(A+90^\circ))
\]

(A sine wave is a cosine wave lagging by \( 90^\circ \))

then:

\[
\cos(B-(A+90^\circ)) = \cos(B)\cos(-A+90^\circ) + \sin(B)\sin(-A+90^\circ)
\]

\[
= \cos(B)\sin(90^\circ-A) + \sin(B)\sin(90^\circ-A)
\]

which depends on:

\[
\cos(90^\circ-A) = \cos(90^\circ)\cos(A) + \sin(90^\circ)\sin(A) = \sin(A) \quad (\cos(90^\circ) = 0, \sin(90^\circ) = 1)
\]

and:

\[
\sin(90^\circ-A) = \cos(90^\circ-A-90^\circ) = \cos(0^\circ-A) = \cos(0^\circ)\cos(A) + \sin(0^\circ)\sin(A) = \cos(A)
\]

\[
(\cos(0^\circ) = 1, \sin(0^\circ) = 0)
\]

therefore:

\[
\sin(A+B) = \cos(B)\sin(A) + \sin(B)\cos(A)
\]

or

\[
\sin(A+B) = \sin(A)\cos(B) + \cos(A)\sin(B)
\]

The others can be found in a similar manner.

\[
\sin(A-B) = \sin(A)\cos(B) - \cos(A)\sin(B)
\]

and

\[
\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)
\]

This leads us on to some of the others

\[
\sin(2A) = \sin(A+A) = \sin(A)\cos(A) + \cos(A)\sin(A) = 2\sin(A)\cos(A)
\]

and

\[
\cos(2A) = \cos(A)\cos(A) - \sin(A)\sin(A) = \cos^2(A) - \sin^2(A)
\]

We can substitute using \( \sin^2(A) + \cos^2(A) = 1 \) to get

\[
\cos(2A) = 1 - 2\sin^2(A) \quad \text{and} \quad \cos(2A) = 2\cos^2(A) - 1
\]
Trigonometric Formulae – as provided in exam formula list

\[
\sin(A \pm B) = \sin(A)\cos(B) \pm \cos(A)\sin(B)
\]

\[
\cos(A \pm B) = \cos(A)\cos(B) \mp \sin(A)\sin(B) \quad \text{- notice the signs!}
\]

\[
\sin(2A) = 2\sin(A)\cos(A)
\]

\[
\cos(2A) = \cos^2(A) - \sin^2(A) \\
= 2\cos^2(A) - 1 \\
= 1 - 2\sin^2(A)
\]

\[
\sin^2(A) + \cos^2(A) = 1
\]

Problems involving the above:

The typical problem involving these formulae is adding waves such as:

\[v_1 = 12\sin(\omega t) \quad \text{and} \quad v_2 = 10\cos(\omega t)\]. Find the wave representing \(v_1 + v_2\) in the form \(R\sin(\omega t + \alpha)\).

\(\omega\) can be any number but it is most likely to be \(n\pi\) where \(n \in \mathbb{N}\) (i.e. \(100\pi\), \(50\pi\) etc).

As long as the frequencies of the waves are identical then the waves can be added in this manner. (Whatever \(\omega\) is doesn’t matter as long as it is still there at the end – it takes no part in the working).

Using \(\sin(A \pm B) = \sin(A)\cos(B) \pm \cos(A)\sin(B)\)

\[R\sin(\omega t + \alpha) = R\sin(\omega t)\cos(\alpha) + R\cos(\omega t)\sin(\alpha)\]

Looking at what we have been given \(R\cos(\alpha) = 12\) and \(R\sin(\alpha) = 10\)

Thinking about a right angled triangle gives us this.

As before, we are looking at rectangular to polar conversion

with \(R^2 = (R\cos(\alpha))^2 + (R\sin(\alpha))^2\) and \(\tan(\alpha) = \frac{R\sin(\alpha)}{R\cos(\alpha)}\).

You could, of course, use your R to P conversion on a calculator to go from \(\begin{pmatrix} R\cos(\alpha) \\ R\sin(\alpha) \end{pmatrix} \) to \(R \angle \alpha\).

This will be the \(R\) and the \(\alpha\) in \(R\sin(\omega t + \alpha)\).

So here we have \(\begin{pmatrix} 12 \\ 10 \end{pmatrix} = 15.62\angle 0.6947\) giving as a resultant wave

\[v_1 + v_2 = 15.62\sin(\omega t + 0.6947)\].
Exercise 3

1. \( V_1 = 50 \sin (150 \pi t) \) and \( V_2 = 70 \cos (150 \pi t) \)

   (a) Calculate the value of \( V_1 + V_2 \) when \( t = 1.7 \)

   (b) Determine \( V_T = V_1 + V_2 \) in the form \( R \sin(\omega t + \alpha) \)

   (c) Calculate \( V_T \) when \( t = 1.7 \).

   Check that your answer agrees with your answer to (a)

2. \( V_1 = 200 \sin (100 \pi t + 0.5) \) and \( V_2 = 40 \sin (100 \pi t + 0.6) \)

   Find \( V_1 + V_2 \) in the form \( R \sin(\omega t + \alpha) \).
Other problems involving multiple angle formulae.

Solve \(3\cos(2t) - 12\cos(t) = -4\) for \(0 \leq t \leq 2\pi\)

There appear to be 2 solutions at \(t \approx 1.5\) and \(t \approx 4.8\) but how do we solve it?

It isn't obvious until we realise that \(\cos(2A) = 2\cos^2(A) - 1\)

This gives us \(3(2\cos^2(t) - 1) - 12\cos(t) = -4\)
\[6\cos^2(t) - 3 - 12\cos(t) = -4\]

Any better? Could you solve \(6x^2 - 12x + 1 = 0\) for \(x\)

If you can then you should be able to solve \(6(\cos(t))^2 - 3 - 12\cos(t) = -4\) for \(\cos(t)\)

\[6\cos^2(t) - 3 - 12\cos(t) = -4\]
\[6\cos^2(t) - 12\cos(t) = -1\]
\[6\cos^2(t) - 12\cos(t) + 1 = 0\]

Use quadratic formula?

\[a = 6, b = -12, c = 1\]

\[\cos(t) = 0.08713\]
\[\cos(t) = 1.9129\]

\(\cos(t) = 0.08713\) obviously can be solved for \(t\) but \(\cos(t) = 1.9129\) can't. This is a common outcome when dealing with expressions of this type.

From the above, when \(\cos(t) = 0.08713\),

\[t = \cos^{-1}(0.08713) = 1.4836\ AND\ t = 2\pi - 1.4836 = 4.7800\]

You should check that these are correct by substituting them into the original expression.
Exercise 4

\[ 6 \cos(t) - 2 \cos(2t) = 1 \quad \text{Solve for } 0 \leq t \leq 2\pi \]

Again, as you can see from the graph, there appear to be 2 solutions in the domain given.
Hyperbolic Functions

The hyperbola generates the hyperbolic sine ($\sinh(x)$) and hyperbolic cosine ($\cosh(x)$) functions but only the hyperbolic cosine function is of any practical use. Shown below is a comparison between a catenary, ($\cosh(x)$), the shape taken up by a cable hanging between two points and a quadratic, the shape taken up by a cable supporting a suspension bridge.

\[
y = \cosh(x) \quad \text{and} \quad y = x^2 + 1
\]

\[
y = \sinh(x)
\]

$\cosh(x)$ and $\sinh(x)$ are asymptotic to $y = x$ and $y = -x$

\[
\begin{array}{|c|c|c|}
\hline
x & \cosh(x) & \sinh(x) \\
\hline
0.1 & 1.0050041681 & 0.10016675 \\
0.5 & 1.1276259652 & 0.52109531 \\
1 & 1.5430806348 & 1.1752011936 \\
2 & 3.7621956911 & 3.6268604078 \\
4 & 27.308232836 & 27.289917197 \\
8 & 1490.4791613 & 1490.4788258 \\
20 & 242582597.7 & 242582597.7 \\
\hline
\end{array}
\]
For the unit circle (radius = 1) you will remember that \((\sin(x))^2 + (\cos(x))^2 = 1\), or \(\sin^2(x) + \cos^2(x) = 1\). The generator for the hyperbolic functions is \(x^2 - y^2 = 1\) which is the equivalent of \((\cosh(x))^2 - (\sinh(x))^2 = 1\) or, as it is usually written

\[ \cosh^2(x) - \sinh^2(x) = 1. \]

But, you can also express the hyperbolic functions as a combination of exponential functions thus:

\( \sinh(x) = \frac{e^x - e^{-x}}{2} \) and \( \cosh(x) = \frac{e^x + e^{-x}}{2} \)

It might be more useful if you were to write them like this:

\[ \sinh(x) = 0.5e^x - 0.5e^{-x} \quad \text{and} \quad \cosh(x) = 0.5e^x + 0.5e^{-x} \]

Notice that these are real number functions - your calculator can be in degrees or radians - you get the same answer. With these facts most problems involving hyperbolic functions can be solved.

**Calculator:** Casio - press the **hyp** button and select from the list

Sharp - **hyp sin, hyp cos** etc..

Interesting properties to explore:

Exercise 5

1. Simplify \( \sinh(x) + \cosh(x) \)

2. Simplify \( \sinh^2(x) + \cosh^2(x) \)

3. Differentiate and simplify \( y = 0.5e^x - 0.5e^{-1x} \)

4. Differentiate and simplify \( y = 0.5e^x + 0.5e^{-1x} \)
Hyperbolic Identities – as provided in exam formula list

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \quad \cosh(x) = \frac{e^x + e^{-x}}{2}$$

Problems involving the above:

Exercise 6

1. Simplify $\sinh^2(x) - \cosh^2(x)$ using the identities given.

2. $\sinh(x) + \cosh(x) = \sqrt{3}$
   Find all the solution for the expression above (solve for $x$).

3. $\sinh(x) \cosh(x) + \frac{1}{4 \exp(2x)} = 5$ Solve for $x$.

4. $y = 0.5 \exp(x) - 0.5 \exp(-x)$
   Find the gradient of $y$ when $x = 3$

5. Find the area under $y = 0.5 \exp(x) - 0.5 \exp(-x)$ between the limits $x = 1$ and $x = 4$. 
Answers – all numerical ones to 4 s.f.

Exercise 1

1. (a) \(-85.65\)  
   (b) 15.04  
   (c) \(-5.062\)

2. (a) \(3 \tan(x)\)  
   (b) \(\sec(10\pi t)\)  
   (c) \(\sin(x)\)

3. (a) 1  
   (b) \(\cos^2(x)\)  
   (c) \(\sin^2(x)\)

Exercise 2

1. 
   \(\frac{7}{\cos(2t)}\)
   \(\frac{7}{\cos(t)}\)
   \(\frac{14 \sin(2t)}{\cos(2t)^2}\)
   \(\text{subst}(1.5, t, (\text{%2}));\)
   \(2.015824978363771\)

2. 
   \(\frac{12}{\tan(2\pi t)}\)
   \(\frac{12}{\tan(2\pi t)}\)
   \(\frac{-24 \pi \sec(2\pi t)^2}{\tan(2\pi t)^2}\)
   \(\text{float}((\text{%3}), \text{numerr});\)
   \(-415.9029916728065\)

Exercise 3

1. (b) \(86.023 \sin(150\pi t + 0.9505)\)  
   (a) and (c) approx -70

2. Try \(200 \leq 0.5 + 40 \leq 0.6\)  
   \(239 \sin(100\pi t + 0.5167)\)

Check: a value for \(t\) in the question should give the same result in the answer.

When \(t=2\), \(V_1 + V_2 = 118.5\)

Exercise 4

\(t = 1.730\) and \(t = 4.553\)

Exercise 5

1. \(e^x\)  
   2. \(\cosh(2x)\)  
   3. \(\frac{dy}{dx} = \cosh(x)\)  
   4. \(\frac{dy}{dx} = \sinh(x)\)

Exercise 6

1. \(-1\)  
   2. \(x = 0.5493\)  
   3. \(x = 1.498\)  
   4. 10.07

5. 25.77